RAMANUJAN'S PARTITION GENERATING FUNCTIONS MODULO ℓ

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ABSTRACT. For the partition function $p(n)$, Ramanujan proved the striking identities

$$\mathcal{P}_5(q) := \sum_{n\geq 0} p(5n+4)q^n = 5\prod_{n\geq 1} \frac{\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty^6},$$

$$\mathcal{P}_7(q) := \sum_{n\geq 0} p(7n+5)q^n = 7\prod_{n\geq 1} \frac{\left(q^7;q^7\right)_\infty^3}{(q;q)_\infty^4} + 49q\prod_{n\geq 1} \frac{\left(q^7;q^7\right)_\infty^7}{(q;q)_\infty^8},$$

where $(q;q)_\infty := \prod_{n\geq 1}(1-q^n)$. As these identities imply his celebrated congruences modulo 5 and 7, it is natural to seek, for primes $\ell \geq 5$, closed form expressions of the power series

$$\mathcal{P}_\ell(q) := \sum_{n\geq 0} p(\ell n - \delta_\ell)q^n \pmod{\ell},$$

where $\delta_\ell := \frac{\ell^2-1}{24}$. In this paper, we prove that

$$\mathcal{P}_\ell(q) \equiv c_\ell \frac{\mathcal{T}_\ell(q)}{\left(q^\ell;q^\ell\right)_\infty} \pmod{\ell},$$

where $c_\ell \in \mathbb{Z}$ is explicit and $\mathcal{T}_\ell(q)$ is the generating function for the Hecke traces of ℓ-ramified values of special Dirichlet series for weight $\ell-1$ cusp forms on $\mathrm{SL}_2(\mathbb{Z})$. This is a new proof of Ramanujan's congruences modulo 5, 7, and 11, as there are no nontrivial cusp forms of weight 4, 6, and 10.

1. Introduction and Statement of Results

A *partition* of n is any nonincreasing sequence of positive integers that sum to n. The number of partitions of n is denoted $p(n)$ (by convention, we let $p(0) := 1$ and $p(n) := 0$ for $n < 0$). Ramanujan famously proved (see [2, 7]), for every non-negative integer n, that

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

For the congruences with modulus 5 and 7, he used the beautiful identities

$$\mathcal{P}_5(q) := \sum_{n\geq 0} p(5n+4)q^n = 5\prod_{n\geq 1} \frac{\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty^6},$$

$$\mathcal{P}_7(q) := \sum_{n\geq 0} p(7n+5)q^n = 7\prod_{n\geq 1} \frac{\left(q^7;q^7\right)_\infty^3}{(q;q)_\infty^4} + 49q\prod_{n\geq 1} \frac{\left(q^7;q^7\right)_\infty^7}{(q;q)_\infty^8},$$

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where $(q;q)_\infty := \prod_{n\geq 1}(1-q^n)$. In 1969, with the help of binary theta functions, Winquist [8] was able to offer another identity that proved Ramanujan's congruence with modulus 11.

In the spirit of these identities, for every prime $\ell \geq 5$, we determine the q-series $\mathcal{P}_\ell(q) \in \mathbb{F}_\ell[[q]]$

$$\mathcal{P}_\ell(q) := \sum_{n\geq 0} p(\ell n - \delta_\ell) q^n \pmod{\ell},$$

where $\delta_\ell := \frac{\ell^2-1}{24}$. These expressions involve the generating functions of "weighted Hecke traces" of special values of specific Dirichlet series associated to weight $\ell - 1$ Hecke eigenforms on $\mathrm{SL}_2(\mathbb{Z})$ (for background see [3] or [6]).

To define these Hecke traces, first suppose that ($q := e^{2\pi i z}$ throughout)

$$f(z) := q + \sum_{n\geq 2} a_f(n) q^n \in S_{2k}$$

is an even integer weight $2k$ Hecke eigenform on $\mathrm{SL}_2(\mathbb{Z})$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2k$, the *twisted quadratic Dirichlet series* is defined by

$$D(f;s) := \sum_{n\geq 1} \frac{\left(\frac{12}{n}\right) a_f\left(\frac{n^2-1}{24}\right)}{n^s},$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. Note that we set $a_f(n) := 0$ if $n \notin \mathbb{Z}$. Furthermore, if $k \geq 2$, $0 \leq j \leq k-2$, and $m \geq 0$, then we let

$$\beta(k,j,m) := \frac{(-1)^{j+1}\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{9}\left(\frac{6}{\pi}\right)^{2k} \frac{(2k+m-2)!(k-j-1)^{[k]}\left(\frac{3}{2}\right)^{[j]}}{j!m!(2k-j-2)!\left(-\frac{1}{2}-j\right)^{[k]}\left(\frac{5}{2}\right)^{[j]}},$$

where $\Gamma(\cdot)$ is the usual Gamma-function. Moreover the *rising factorial* is given by

$$(x)^{[j]} := \begin{cases} x(x+1)\cdots(x+j-1) & \text{if } j \geq 1, \\ 1 & \text{if } j = 0, \end{cases}$$

which are companions of the usual *falling factorials*

$$(x)_m := \begin{cases} x(x-1)\cdots(x-m+1) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0, \\ \frac{1}{(x)_{-m}} & \text{if } m \leq -1. \end{cases}$$

For such $f \in S_{2k}$, we define the following sums of values of Dirichlet series by[1]

$$D_f := \sum_{j=0}^{k-2} \sum_{m\geq 0} \beta(k,j,m) D(f; 2k+1+2m+2j).$$

Moreover we define, for $n \in \mathbb{N}$, the *weight $2k$ Hecke trace* by

$$\mathrm{Tr}_{2k}(n) := \sum_f a_f(n) \frac{D_f}{||f||},$$

[1]Convergence can be concluded from Theorem 1.4 of [4].

where the sum runs over the normalized Hecke eigenforms $f \in S_{2k}$, and the Petersson norms of f, $||f||$, is defined as ($z = x + iy$ throughout)

$$||f|| := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |f(z)|^2 y^{2k} \frac{dxdy}{y^2}.$$

As $a_f(n)$ is the eigenvalue of the Hecke operator T_n, we refer to the numbers $\mathrm{Tr}_{2k}(n)$ as Hecke traces. Finally, for primes $\ell \geq 5$, we collect the ℓ-ramified values (i.e., the arguments that are multiples of ℓ) if $2k = \ell - 1$ as the Fourier coefficients of the generating function

$$\mathcal{T}_\ell(q) := \sum_{n \geq 1} \mathrm{Tr}_{\ell-1}(\ell n) q^n.$$

Theorem 1.1. *If $\ell \geq 5$ is a prime, then*

$$\mathcal{P}_\ell(q) \equiv c_\ell \frac{\mathcal{T}_\ell(q)}{(q^\ell; q^\ell)_\infty} \pmod{\ell},$$

where $c_\ell := 2 \cdot \overline{3}(\frac{-1}{\ell})(\frac{\ell+1}{2})!^{\ell-3} \pmod{\ell}$, where throughout $\overline{a}$ denotes the inverse of $a \pmod{\ell}$ and where $(\frac{\cdot}{\cdot})$ denotes the Kronecker symbol.

For $\ell \in \{5, 7, 11\}$, we have that $S_{\ell-1} = \{0\}$. As there are no nontrivial cusp forms in these spaces, we immediately obtain a new proof of Ramanujan's famous partition congruences.

Corollary 1.2. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Moreover Theorem 1.1 immediately implies the following congruence formula for $p(\ell n - \delta_\ell) \pmod{\ell}$ in terms of $p(0), p(1), \ldots, p(n-1)$.

Corollary 1.3. *If $\ell \geq 5$ is a prime and $n \in \mathbb{N}$, then we have*

$$p(\ell n - \delta_\ell) \equiv c_\ell \sum_{\substack{j,m \geq 0 \\ \ell j + m = n}} p(j) \, \mathrm{Tr}_{\ell-1}(\ell m) \pmod{\ell}.$$

Example. For the prime $\ell = 13$, Theorem 1.4 and Corollary 1.3 of [4] gives

$$\mathcal{T}_{13}(q) = -\frac{33108590592}{691} \Delta | U_{13}(z) \equiv 7\Delta | U_{13}(z) \pmod{13},$$

where $f|U_j(z) := \sum_{n \geq 1} a_f(jn) q^n$ for $j \in \mathbb{N}$. Using $c_{13} \equiv 6 \pmod{13}$, we obtain

$$c_{13} \frac{\mathcal{T}_{13}(q)}{(q^{13}; q^{13})_\infty} \equiv \frac{3\Delta|U_{13}(z)}{(q^{13}; q^{13})_\infty} \equiv 11q + 9q^2 + 3q^3 + 6q^4 + 12q^5 + 6q^6 + q^8 + \ldots \pmod{13}.$$

To illustrate Theorem 1.1, we note that

$$\begin{aligned} \mathcal{P}_{13}(q) = \sum_{n \geq 1} p(13n - 7) q^n &= 11q + 490q^2 + 8349q^3 + 89134q^4 + 715220q^5 + \ldots \\ &\equiv 11q + 9q^2 + 3q^3 + 6q^4 + 12q^5 + 6q^6 + q^8 + \ldots \pmod{13}. \end{aligned}$$

Furthermore, Corollary 1.3 implies, for $n \in \mathbb{N}$, that

$$p(13n - 7) \equiv 3 \sum_{\substack{j,m \geq 0 \\ 13j + m = n}} p(j) \tau(13m) \pmod{13}.$$

To obtain Theorem 1.1, we make use of recent work of Gomez, the third author, Saad, and Singh [4] that offers an infinite family of generalizations of Euler's "Pentagonal Number" recurrence for $p(n)$. In Section 2 we recall these formulas, and in Section 3 we use them to obtain Theorem 1.1.

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2. Generalizations of Euler's "Pentagonal number" recurrence

For $n \in \mathbb{N}$, Euler's famous recurrence relation asserts that (see p. 12 of [1])

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots = \sum_{m \in \mathbb{Z} \setminus \{0\}} (-1)^{m+1} p(n - \omega(m)), \tag{2.1}$$

where $\omega(m) := \frac{3m^2+m}{2}$ is the m-th *pentagonal number*. This recurrence is one of the most efficient methods for computing partition numbers.

Gomez, the third author, Saad, and Singh [4] proved that Euler's recurrence is the first case of an infinite family of rich recurrence relations satisfied by the partition numbers. To make this precise, we make use of *Dedekind's eta-function*

$$\eta(z) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{24}(6n+1)^2},$$

where $z \in \mathbb{H}$, the upper half of the complex plane. To define these relations, we require the differential operator $D := \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}$. For $k \in \mathbb{N}_0$, we define[2] [3]

$$R_k(z) := \frac{(2k-1)(2k-2)_{k-1}^2}{2^{2k-2}} \sum_{\substack{r,s \geq 0 \\ r+s=k}} (-1)^{r+1} \frac{2s-1}{(2r)!(2s)!} D^r \left(\frac{1}{\eta(z)} \right) D^s(\eta(z)).$$

By [4], we have

$$R_k(z) = \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} (-1)^{m+1} g_k(n,m) p(n - \omega(m)) q^n,$$

where

$$g_k(n,m) := \frac{(2k-1)(2k-2)_{k-1}^2}{2^{2k-2}} \sum_{r=0}^{k} (-1)^{k+r} \frac{2k-2r-1}{(2r)!(2k-2r)!} (6m+1)^{2r} \left(24n - (6m+1)^2\right)^{k-r}.$$

[2] To avoid confusing notation, we note that $R_k(z)$ is denoted $P_k(z)$ in [4].

[3] We note a small typographical error in [4] (there the $(-1)^{r+1}$ is $(-1)^r$).

By Theorem 1.1 of [4], for each $k \geq 0$, R_k is a weight $2k$ holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$. These expressions are simple to compute for $k \leq 13$ apart from $k = 12$. Namely, Corollaries 1.2 and 1.3 of [4] give the following identities in terms of the usual Eisenstein series

$$E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n,$$

where B_r denotes the r-th Bernoulli number, $\sigma_r(n) := \sum_{d|n} d^r$ the *r-th divisor sum*, and $\Delta(z) := \eta^{24}(z)$.

Theorem 2.1. *The following are true:*

(1) *If $k \in \{0, 1\}$, then we have*

$$R_k(z) = \begin{cases} -1 & \text{if } k = 0, \\ 0 & \text{if } k = 1. \end{cases}$$

(2) *If $k \in \{2, 3, 4, 5, 7\}$, then we have*

$$R_k(z) = \binom{2k-2}{k-2} E_{2k}(z).$$

(3) *If $k \in \{6, 8, 9, 10, 11, 13\}$, then we have*

$$R_k(z) = \binom{2k-2}{k-2} E_{2k}(z) + \beta_k \Delta_{2k}(z),$$

where

$$\Delta_{2k}(z) := q + \sum_{n \geq 2} \tau_{2k}(n) q^n := \begin{cases} \Delta(z) & \text{if } k = 6, \\ \Delta(z) E_4(z) & \text{if } k = 8, \\ \Delta(z) E_6(z) & \text{if } k = 9, \\ \Delta(z) E_4^2(z) & \text{if } k = 10, \\ \Delta(z) E_4(z) E_6(z) & \text{if } k = 11, \\ \Delta(z) E_4^2(z) E_6(z) & \text{if } k = 13, \end{cases}$$

where we let

$$\beta_k := \begin{cases} -\frac{33108590592}{691} & \text{if } k = 6, \\ -\frac{187167592415232}{3617} & \text{if } k = 8, \\ -\frac{28682634201661440}{43867} & \text{if } k = 9, \\ -\frac{8294726176465158144}{174611} & \text{if } k = 10, \\ -\frac{1014750650737345167736}{77683} & \text{if } k = 11, \\ -\frac{11950657342663397002444 80}{657931} & \text{if } k = 13. \end{cases}$$

Finally, for general k, Theorem 1.4 of [4] gives the following expressions that make use of the weighted Hecke trace generating function

$$T_{2k}(z) := \sum_{n \geq 1} \mathrm{Tr}_{2k}(n) q^n \in S_{2k}.$$

Theorem 2.2. *If $k \geq 6$, with $k \neq 7$, then we have*

$$R_k(z) = \binom{2k-2}{k-2} E_{2k}(z) + T_{2k}(z).$$

These results are equivalent to the infinite family of recurrence relations given in the following corollary.

Corollary 2.3. *If n is a positive integer, then the following are true:*
(1) *We have that*

$$p(n) = \sum_{m \in \mathbb{Z} \setminus \{0\}} (-1)^{m+1} p\left(n - \omega(m)\right).$$

(2) *If $k \in \{2,3,4,5,7\}$, then we have*

$$p(n) = \frac{1}{g_k(n,0)} \left(-\frac{4k}{B_{2k}} \binom{2k-2}{k-2} \sigma_{2k-1}(n) + \sum_{m \in \mathbb{Z} \setminus \{0\}} (-1)^{m+1} g_k(n,m)\, p(n - \omega(m)) \right).$$

(3) *If $k \in \{6,8,9,10,11,13\}$, then we have*

$$p(n) = \frac{1}{g_k(n,0)} \left(-\frac{4k}{B_{2k}} \binom{2k-2}{k-2} \sigma_{2k-1}(n) + \beta_k \tau_{2k}(n) + \sum_{m \in \mathbb{Z} \setminus \{0\}} (-1)^{m+1} g_k(n,m)\, p(n - \omega(m)) \right).$$

(4) *If $k \geq 6$, with $k \neq 7$, then we have*

$$p(n) = \frac{1}{g_k\left(n,0\right)} \left(-\frac{4k}{B_{2k}} \binom{2k-2}{k-2} \sigma_{2k-1}(n) + \mathrm{Tr}_{2k}(n) + \sum_{m \in \mathbb{Z} \setminus \{0\}} (-1)^{m+1}\, g_k\left(n,m\right) p\left(n - \omega(m)\right) \right).$$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 requires the following elementary proposition regarding the congruence properties of certain examples of Corollary 2.3 (4). Namely, we obtain a pentagonal number recurrence modulo ℓ for the Hecke traces with argument ℓn, where the pentagonal numbers $\omega(m)$ are restricted to a fixed congruence class modulo ℓ.

Proposition 3.1. *If $\ell \geq 5$ is prime and n is a positive integer, then*

$$\mathrm{Tr}_{\ell-1}\left(\ell n\right) \equiv -3 \cdot \overline{2} \left(\frac{\ell+1}{2}\right)!^2 \sum_{\substack{m \in \mathbb{Z} \\ 6m \equiv -1 \pmod{\ell}}} (-1)^{m+1} p\left(\ell n - \omega(m)\right) \pmod{\ell}.$$

Proof. By Corollary 2.3 (4), we have, for $k \geq 2$, that

$$p(n) = \frac{1}{g_k\left(n,0\right)} \left(-\frac{4k}{B_{2k}} \binom{2k-2}{k-2} \sigma_{2k-1}(n) + \mathrm{Tr}_{2k}(n) + \sum_{m \in \mathbb{Z} \setminus \{0\}} (-1)^{m+1}\, g_k\left(n,m\right) p\left(n - \omega(m)\right) \right).$$

By letting $k = \frac{\ell-1}{2}$, the von Stadt–Clausen Theorem (for example, see [5, Theorem 3, pg. 233]) implies that the denominator of the Bernoulli number $B_{\ell-1}$ is divisible by ℓ, which in turn implies that the divisor function contribution above vanishing modulo ℓ. By then letting $n \mapsto \ell n$, we obtain

$$p\left(\ell n\right) \equiv \frac{1}{g_{\frac{\ell-1}{2}}\left(\ell n,0\right)} \left(\mathrm{Tr}_{\ell-1}\left(\ell n\right) + \sum_{m \in \mathbb{Z} \setminus \{0\}} (-1)^{m+1}\, g_{\frac{\ell-1}{2}}\left(\ell n,m\right) p\left(\ell n - \omega(m)\right) \right) \pmod{\ell}. \tag{3.1}$$

By direct calculation, we have

$$g_{\frac{\ell-1}{2}}(\ell n,m)=\frac{(\ell-2)\,(\ell-3)^2_{\frac{\ell-3}{2}}}{2^{\ell-3}}\sum_{r=0}^{\frac{\ell-1}{2}}(-1)^{\frac{\ell-1}{2}+r}\frac{\ell-2-2r}{(2r)!\,(\ell-1-2r)!}(6m+1)^{2r}\left(24\ell n-(6m+1)^2\right)^{\frac{\ell-1}{2}-r}$$

$$\equiv\frac{16}{2^\ell}(\ell-3)^2_{\frac{\ell-3}{2}}(6m+1)^{\ell-1}\sum_{r=0}^{\frac{\ell-1}{2}}\frac{2r+2}{(2r)!\,(\ell-1-2r)!}\equiv\frac{32}{2^\ell}(\ell-3)^2_{\frac{\ell-3}{2}}(6m+1)^{\ell-1}\sum_{r=0}^{\frac{\ell-1}{2}}\binom{\ell-1}{2r}\frac{r+1}{(\ell-1)!}$$

$$\equiv\varrho_\ell(6m+1)^{\ell-1}\equiv\begin{cases}\varrho_\ell & m\not\equiv-\overline{6},\\ 0 & m\equiv-\overline{6},\end{cases}\pmod{\ell},\tag{3.2}$$

where

$$\varrho_\ell:=\frac{32}{2^\ell}(\ell-3)^2_{\frac{\ell-3}{2}}\sum_{r=0}^{\frac{\ell-1}{2}}\binom{\ell-1}{2r}\frac{r+1}{(\ell-1)!}.\tag{3.3}$$

To compute ϱ_ℓ, we note that for $M\geq1$, we have

$$\sum_{r=0}^{M}\binom{2M}{2r}r=2^{2M-2}M\quad\text{and}\quad\sum_{r=0}^{M}\binom{2M}{2r}=2^{2M-1}.$$

Therefore, by setting $M=\frac{\ell-1}{2}$, we have

$$\sum_{r=0}^{\frac{\ell-1}{2}}\binom{\ell-1}{2r}(r+1)\equiv\frac{\ell-1}{2}2^{\ell-3}+2^{\ell-2}=3\cdot2^{\ell-4}\pmod{\ell}.$$

Combining this with (3.3), we obtain

$$\varrho_\ell\equiv\frac{6\,(\ell-3)^2_{\frac{\ell-3}{2}}}{(\ell-1)!}\pmod{\ell}.$$

After application of Wilson's Theorem we see that

$$\varrho_\ell\equiv-6\,(\ell-3)^2_{\frac{\ell-3}{2}}\pmod{\ell}.$$

Finally, we note that

$$(\ell-3)_{\frac{\ell-3}{2}}\equiv\frac{(-1)^{\frac{\ell-3}{2}}\left(\frac{\ell+1}{2}\right)!}{2}\pmod{\ell}.$$

Thus

$$\varrho_\ell\equiv-6\left(\frac{(-1)^{\frac{\ell-3}{2}}\left(\frac{\ell+1}{2}\right)!}{2}\right)^2\equiv-3\cdot\overline{2}\left(\frac{\ell+1}{2}\right)!^2\pmod{\ell}.\tag{3.4}$$

Therefore, we have by (3.1) and (3.2)

$$\begin{aligned}\varrho_\ell p(\ell n)&\equiv\mathrm{Tr}_{\ell-1}(\ell n)+\varrho_\ell\sum_{m\in\mathbb{Z}\setminus\{0\}}(-1)^{m+1}(6m+1)^{\ell-1}p(\ell n-\omega(m))\\&\equiv\mathrm{Tr}_{\ell-1}(\ell n)+\varrho_\ell\sum_{\substack{m\in\mathbb{Z}\setminus\{0\}\\6m\not\equiv-1\pmod{\ell}}}(-1)^{m+1}p(\ell n-\omega(m))\pmod{\ell}.\end{aligned}\tag{3.5}$$

Now, substituting $n \mapsto \ell n$ in (2.1) and multiplying by ϱ_ℓ on both sides gives

$$\varrho_\ell p(\ell n) \equiv \varrho_\ell \sum_{m\in\mathbb{Z}\setminus\{0\}} (-1)^{m+1} p(\ell n - \omega(m))\, q^n \pmod{\ell}.$$

By subtracting (3.5) from this on both sides, we obtain

$$0 \equiv -\mathrm{Tr}_{\ell-1}(\ell n) + \varrho_\ell \sum_{\substack{m\in\mathbb{Z}\\ 6m\equiv -1 \pmod{\ell}}} (-1)^{m+1} p(\ell n - \omega(m)) \pmod{\ell}.$$

Solving for $\mathrm{Tr}_{\ell-1}(\ell n)$ and substituting (3.4) gives the claim. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Proposition 3.1 is equivalent to the generating function congruence

$$\mathcal{T}_\ell(q) \equiv -3\cdot \overline{2}\left(\frac{\ell+1}{2}\right)!^2 \sum_{n\geq 0} \sum_{\substack{m\in\mathbb{Z}\\ \omega(m)\equiv\delta_\ell \pmod{\ell}}} (-1)^{m+1} p(\ell n - \omega(m))\, q^n \pmod{\ell},$$

where we note that $6m \equiv -1 \pmod{\ell}$ is equivalent to $\omega(m) \equiv \delta_\ell \pmod{\ell}$. By taking a convolution product, we see that

$$\sum_{n\geq 0} \sum_{\substack{m\in\mathbb{Z}\\ \omega(m)\equiv\delta_\ell \pmod{\ell}}} (-1)^{m+1} p(\ell n - \omega(m))\, q^n \equiv \mathcal{P}_\ell(q)\,\theta_\ell(q) \pmod{\ell},$$

for some q-series

$$\theta_\ell(q) := \sum_{s\in\mathbb{Z}} (-1)^{y_\ell(s)} q^{w_\ell(s)}.$$

We now turn to the explicit calculation of $\theta_\ell(q)$, which then completes the proof. To this end, we observe that the n-th Fourier coefficient of $\mathcal{P}_\ell(q)\,\theta_\ell(q)$ is

$$\sum_{\substack{m\in\mathbb{Z}\\ \omega(m)\equiv\delta_\ell \pmod{\ell}}} (-1)^{m+1} p(\ell n - \omega(m)) \equiv \sum_{s\in\mathbb{Z}} (-1)^{y_\ell(s)} p(\ell n - (\ell w_\ell(s) + \delta_\ell)) \pmod{\ell}.$$

To identify $w_\ell(s)$, we solve $\ell w_\ell(s) + \delta_\ell = \omega(m)$ for $m \equiv -\overline{6} \pmod{\ell}$. Now, define α_ℓ by $6\alpha_\ell = \ell m_\ell - 1$ with $m_\ell = \pm 1$ chosen so that $\alpha_\ell = \frac{\ell m_\ell - 1}{6} \in \mathbb{Z}$. Then by setting $m = \ell s + \alpha_\ell$ in the formula for $\omega(m)$ and simplifying, we see that

$$\omega(\ell s + \alpha_\ell) = \ell\frac{3\ell s^2 + 6\alpha_\ell s + s}{2} + \frac{3\alpha_\ell^2 + \alpha_\ell}{2} = \ell\frac{3\ell s^2 + \ell m_\ell s}{2} + \delta_\ell.$$

Thus

$$w_\ell(s) = \frac{3\ell s^2 + \ell m_\ell s}{2} = \begin{cases} \dfrac{3\ell s^2 + \ell s}{2} & \text{if } \ell \equiv 1 \pmod{6}, \\[2ex] \dfrac{3\ell s^2 - \ell s}{2} & \text{if } \ell \equiv 5 \pmod{6}. \end{cases}$$

Likewise, by comparing $(-1)^{y_\ell(s)} = (-1)^{m+1}$ if $m = \ell s + \alpha_\ell$ with the same choice of α_ℓ, we can set $y_\ell(s) = s + \alpha_\ell + 1$. We therefore obtain after some calculation that for $\ell \equiv 1 \pmod{6}$, we have, using

(3.1),

$$\theta_\ell(q)=\sum_{s\in\mathbb{Z}}(-1)^{s+\frac{\ell-1}{6}+1}\,q^{\frac{3s^2+s}{2}\ell}=(-1)^{\frac{\ell-1}{6}+1}\sum_{s\in\mathbb{Z}}(-1)^{s}\,q^{\frac{3s^2+s}{2}\ell}=(-1)^{\frac{\ell+5}{6}}\left(q^\ell;q^\ell\right)_\infty .$$

Likewise for $\ell\equiv 5\pmod 6$ we have, using (3.1),

$$\theta_\ell(q)=\sum_{s\in\mathbb{Z}}(-1)^{s+\frac{\ell+1}{6}+1}\,q^{\frac{3s^2-s}{2}\ell}=(-1)^{\frac{\ell+1}{6}}\sum_{s\in\mathbb{Z}}(-1)^{s+1}\,q^{\frac{3s^2-s}{2}\ell}=(-1)^{\frac{\ell-1}{6}}\left(q^\ell;q^\ell\right)_\infty .$$

Now note that

$$-\left(\frac{-1}{\ell}\right)=\begin{cases}(-1)^{\frac{\ell+5}{6}} & \text{if } \ell\equiv 1\pmod 6,\\ (-1)^{\frac{\ell+1}{6}} & \text{if } \ell\equiv 5\pmod 6.\end{cases}$$

We conclude

$$c_\ell\equiv-\left(\frac{-1}{\ell}\right)\overline{-3\cdot\overline{2}\left(\frac{\ell+1}{2}\right)!^2}\equiv 2\cdot\overline{3}\left(\frac{-1}{\ell}\right)\left(\frac{\ell+1}{2}\right)!^{\ell-3}\pmod\ell,$$

which completes the proof. □

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